The probability of exceeding a piecewise deterministic barrier by the heavy-tailed renewal compound process

Zbigniew Palmowski* Martijn Pistorius†

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Abstract

We analyze the asymptotics of crossing a high piecewise linear barriers by a renewal compound process with the subexponential jumps. The study is motivated by ruin probabilities of two insurance companies (or two branches of the same company) that divide between them both claims and premia in some specified proportions when the initial reserves of both companies tend to infinity.

Key words: First time passage problem, ruin probabilities, subexponential distribution.

*University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland, E-mail: zpalma@math.uni.wroc.pl

†Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK, E-mail:Martijn.Pistorius@kcl.ac.uk

1 Motivation

The study of boundary crossing probabilities of a stochastic process with heavy tailed increments has applications in fields such as queuing theory, insurance and finance. In this paper we consider the problem of a renewal process crossing a high piecewise linear boundary. This study is in particular motivated by ruin probabilities of two insurance companies with proportional claims (see [1]) and the steady state distribution of a tandem queue with two servers (see [**?**]). To be more precise, we set

$$S_t = \sum_{i=1}^{N_t} \sigma_i \tag{1}$$

for N_t a renewal process with i.i.d. inter-arrival times ζ_i, and the claims σ_i are i.i.d. random variables independent of $N(t)$, with the distribution function $F(x)$. We shall denote by λ and μ the reciprocals of the means of ζ_i and σ_i, respectively. Let the boundaries b_1, b_2 given by

$$b_1(t) = b_1(t; x_1) = x_1 + p_1 t, \qquad b_2(t) = b_2(t; x_2) = x_2 + p_2 t,$$

where we assume that

$$p_1 > p_2, \qquad p_2 > \rho := \frac{\lambda}{\mu} = E[\sigma]/E[\zeta] \tag{2}$$

and consider the following boundary crossing probabilities:

$$\begin{aligned}
\psi_\wedge(x_1, x_2) &= P(\exists t \geq 0 : S_t > b_1(t) \wedge b_2(t)) \\
\psi_\vee(x_1, x_2) &= P(\exists t \geq 0 : S_t > b_1(t) \vee b_2(t)) \\
\psi_\times(x_1, x_2) &= P(\exists t \geq 0, u \geq 0 : S_t > b_1(t), S_u > b_2(u)),
\end{aligned}$$

where $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

The $\psi_\wedge(x_1, x_2)$ describes the ruin probability of at least one insurance company. $\psi_\vee(x_1, x_2)$ corresponds to the simultaneous ruin of the insurance companies. Finally, $\psi_\times(x_1, x_2)$ describes probability that both companies will have ruin. First assumption in (2) means that the second company receives less premium per amount paid out and second one is the stability condition under which reserves of both insurance companies tend to infinity. The solutions of the "degenerate two-dimensional" ruin problems strongly depend on the relative position of the vector of premium rates $p = (p_1, p_2)$ with respect to the proportions vector $(1, 1)$. Namely, if the initial reserves satisfy $x_2 \leq x_1$, the two lines do not intersect. It follows therefore that the barriers the "$\wedge$" and "$\vee$" ruin always happen for the second and first company respectively. In this case the asymptotics follows from one-dimensional ruin theory – see e.g. Rolski et al. (1999). Therefore we focus here on the opposite case when $x_1 < x_2$.

In this paper we derive the exact first order asymptotics of these ruin probabilities if x_1, x_2 tend to infinity according to a ray in the positive quadrant (i.e. x_1/x_2 is constant) if the claims follow a subexponential distribution. We model

the claims by subexponential distributions since many catastrophic events like earthquakes, storms, terrorist attacks etc are used in their description. Insurance companies use e.g. the lognormal distribution (which is subexponential) to model car claims – see Rolski et al. (1999) or Embrechts et al. (1997) for the further background.

2 Main result

In order to state our results we start with recalling some notions. A distribution function G on $[0,\infty)$ is subexponential ($G \in \mathcal{S}$) if

$$\lim_{x\to\infty} \overline{G^{*2}}(x)/\overline{G}(x) = 2 \,,$$

where G^{*2} is the fold of G with itself. The integrated tail distribution G_I of G is defined as $G_I(x) = \nu^{-1}\int_0^x \overline{G}(y)\mathrm{d}y$, where $\nu = \int_0^\infty \overline{G}(x)\,\mathrm{d}x$. In fact we will impose the stronger condition, that is,

$$\overline{G}(x) \sim Ax^{-\alpha}, \quad 1 < \alpha < 2. \tag{3}$$

Note that G is long-tailed:

$$\overline{G}(x) > 0 \text{ for all } x, \qquad \lim_{x\to\infty} \frac{\overline{G}(x-h)}{\overline{G}(x)} = 1, \quad \text{for all fixed } h > 0. \tag{4}$$

Let F denote the distribution of the claim size σ and define

$$H(aK,K) = \int_0^\infty \overline{F}\left(\max\{aK + m_1 t, K + m_2 t\}\right)\,\mathrm{d}t \tag{5}$$

and

$$J(aK,K) = \int_0^\infty \overline{F}\left(\min\{aK + m_1 t, K + m_2 t\}\right)\,\mathrm{d}t \tag{6}$$

for $m_i = p_i E[\zeta] - E[\sigma]$ $(i = 1,2)$.

In view of (2) we note that if $a \geq 1$, $b_1 \vee b_2 = b_1$ and $b_1 \wedge b_2 = b_2$, so that the crossing problems are one-dimensional. If F_I follows a subexponential distribution, Veraverbeke's theorem (see Veraverbeke (1977), Embrechts and Veraverbeke (1982) and also Zachary (2004) for a short proof) implies that

$$\psi_\wedge(aK,K) \sim \frac{1}{m_2\mu}\overline{F}_I(K), \quad \psi_\vee(aK,K) \sim \frac{1}{m_1\mu}\overline{F}_I(aK), \quad K \to \infty,$$

where we write $f(x) \sim g(x)$ $(x \to \infty)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. In the opposite case that $a < 1$ the asymptotics of $\psi_\wedge/\psi_\vee/\psi_\times$ are as follows:

Theorem 1 *Let $a < 1$. Assume that $E[\zeta^3] < \infty$. If F satisfies (3) with $1 < \alpha < 2$, then it holds that, as $K \to \infty$,*

$$\psi_\wedge(aK,K) \sim \frac{1}{m_1\mu}\overline{F}_I(aK) + \frac{1}{m_2\mu}\overline{F}_I(K) - H(aK,K) \sim J(aK,K)\,, \tag{7}$$

$$\psi_\vee(aK,K) \sim \psi_\times(aK,K) \sim H(aK,K). \tag{8}$$

It is worth observing that, in contrast to the case of light-tails (see [1, Thm. 1]), the asymptotic probability of crossing both b_1 and b_2 at the same time appears in the asymptotics for $\psi_\wedge(aK, K)$ (and is equal to $H(aK, K)$). In the setting of (re)insurance companies with proportional claims (see [1] for details) this result agrees with what we might expect - "large" claims cause often bankruptcy not only of the insurance company but of a whole chain of reinsurers.

The proof of the main result follows from the Lemmas ... The paper is organized as follows. In Section 3 we present useful facts and in Section 4 main (lower and upper) bounds giving the Theorem 1.

3 Key Lemmas

Let $T_0 = 0, T_n = \sum_{i=1}^n \zeta_i$ and $\Xi_0 = 0, \Xi_n = \sum_{i=1}^n \sigma_i$ denote the two random walks of claims and interarrival times and define the associated random walks $(S_n^{(1)})_{n\geq 0}$ and $(S_n^{(2)})_{n\geq 0}$ by

$$S_n^{(i)} = \Xi_n - p_i T_n \quad i = 1, 2.$$

Let $M_n^{(i)} = \max_{m\leq n} S_m^{(i)}$ and $M_{[n,\infty)}^{(i)} = \max_{m\geq n} S_m^{(i)}$ with $M_\infty^{(i)} = M_{[0,\infty)}^{(i)}$. Also let

$$T = \frac{x_2 - x_1}{m_1 - m_2} = \frac{x_2 - x_1}{(p_1 - p_2)E[\zeta]}$$

be the epoch that the lines $\widetilde{b}_1(t) = x_1 + m_1 t$ and $\widetilde{b}_2(t) = x_2 + m_2 t$ cross and let

$$\widetilde{T} = \frac{x_2 - x_1}{p_1 - p_2} = \frac{T}{\lambda}$$

be the epoch that the lines $b_1(t) = x_1 + p_1 t$ and $b_2(t) = x_2 + p_2 t$ cross. We denote

$$c = \frac{1 - a}{m_1 - m_2}$$

and note that $T = cK$ if $x_1 = aK$ and $x_2 = K$. Define

$$N^* = \min\{n \geq 1 : T_n > \widetilde{T}\} - 1$$

and note that N^* is a stopping time w.r.t. $\sigma(\{\Xi_n, T_n\}_n)$. Further, in view of the strong law of large numbers $N^*/T \to 1$. From the Theorem V.5.14 of Petrov (1972) we have the following upper bound:

Lemma 1 *Let $\zeta^3 < \infty$. Denote $\rho = E|\zeta - E\zeta|^3/s^2$ for $s^2 = E(\zeta - E\zeta)^2$. Then*

$$\left| P\left(\frac{T_n - nE\zeta}{\sqrt{n}} < x\right) - \Phi(x)\right| \leq A \frac{\rho}{\sqrt{n}(1 + |x|)^3}$$

for all x and fixed A.

Now, we have

$$P(N^* > (1+\epsilon)T) = P(T_{[(1+\epsilon)T]} < \tilde{T})$$
$$= P\left(\frac{T_{[(1+\epsilon)T]} - (1+\epsilon)T\frac{1}{\lambda}}{\sqrt{T(1+\epsilon)}} < -\sqrt{T}\frac{\epsilon}{\lambda\sqrt{(1+\epsilon)}}\right) \le \Phi\left(-\sqrt{T}\frac{\epsilon}{\lambda\sqrt{(1+\epsilon)}}\right)$$
$$+A\frac{\rho(1+\epsilon)\lambda^3}{T^2\epsilon^3} = \mathrm{O}(T^{-2}) = o(T^{-\alpha}).$$

Hence

$$P(N^* > (1+\epsilon)T) = \mathrm{o}(\overline{F}(T)), \tag{9}$$

Similarly,

$$P(N^* < (1-\epsilon)T) = \mathrm{o}(\overline{F}(T)). \tag{10}$$

and

$$\begin{aligned} P(T_T < (1-\epsilon)E\zeta T) &= \mathrm{o}(\overline{F}(T)), &(11)\\ P(T_T > (1+\epsilon)E\zeta T) &= \mathrm{o}(\overline{F}(T)). &(12)\end{aligned}$$

A key step is to note that the ruin probabilities can be defined in terms of these quantities as follows:

Proposition 1 *It holds that*

$$\begin{aligned}
\psi_\vee(aK,K) &= P(M^{(2)}_{N^*} > K) + P(M^{(2)}_{N^*} \le K, M^{(1)}_{[N^*+1,\infty)} > aK),\\
\psi_\wedge(aK,K) &= P(M^{(1)}_{N^*} > aK) + P(M^{(1)}_{N^*} \le aK, M^{(2)}_{[N^*+1,\infty)} > K),\\
\psi_\times(aK,K) &= P(M^{(2)}_{\infty} > K) - P(M^{(2)}_{\infty} > K, M^{(1)}_{\infty} \le aK)\\
&= P(M^{(2)}_{\infty} > K) + P(M^{(1)}_{N^*} \le aK, M^{(1)}_{[N^*+1,\infty)} > aK)\\
&\quad -P(M^{(1)}_{N^*} \le aK, M^{(2)}_{[N^*+1,\infty)} > K).
\end{aligned}$$

Indeed note that

$$P(M^{(2)}_{\infty} > K, M^{(1)}_{\infty} \le aK) = P(\max\{M^{(2)}_{N^*}, M^{(2)}_{[N^*+1,\infty)}\} > K, M^{(1)}_{N^*} \le aK, M^{(1)}_{[N^*+1,\infty)} \le aK)$$
$$= P(M^{(2)}_{[N^*+1,\infty)} > K, M^{(1)}_{N^*} \le aK, M^{(1)}_{[N^*+1,\infty)} \le aK)$$
$$= P(M^{(1)}_{N^*} \le aK, M^{(2)}_{[N^*+1,\infty)} > K) - P(M^{(1)}_{N^*} \le aK, M^{(1)}_{[N^*+1,\infty)} > aK).$$

Note that in view of Veraverbeke's theorem

$$P(M^{(2)}_{\infty} > K) \sim \frac{1}{m_2\mu}\overline{F}_I(K).$$

To prove the Theorem 1 we will estimate the other terms.

4 Main estimates

Before we proceed we first introduce some extra notation and collect some auxiliary results. Let

$$H(x,y) = I^{(2)}_{[0,T]}(x) + I^{(1)}_{[T,\infty]}(y), \tag{13}$$
$$J(x,y) = I^{(1)}_{[0,T]}(x) + I^{(2)}_{[T,\infty]}(y), \tag{14}$$

where $I^{(i)}_{[x,y]}(w) = \int_x^y \overline{F}\,(w+m_i t)\ \mathrm{d}t$ for $i=1,2$. Note that definition of $H(ak,k)$ and $J(aK,K)$ agree with (5) and (6) respectively.

The following result relates $H(aK,K)$ to $I^{(2)}_{[0,\infty]}(aK)$ and $J(aK,K)$ to $I^{(1)}_{[0,\infty]}(K)$:

Lemma 1 *The following hold true:*

(i) $I^{(1)}_{[T,\infty]}(aK) = (m_2/m_1) I^{(2)}_{[T,\infty]}(K)$,

(ii) $H(aK,K) \le I^{(2)}_{[0,\infty]}(aK)$.

(iii) $J(aK,K) \le (m_1/m_2) I^{(1)}_{[0,\infty]}(aK)$,

(iv) For $i=1,2$ *it holds that* $I^{(i)}_{[\alpha,\beta]}(\zeta) \le \frac{\beta-\alpha}{\alpha} I^{(i)}_{[0,\alpha]}(\zeta)$ *where* $\alpha,\beta,\zeta>0$ *are constants with* $\alpha<\beta$. *In particular,* $I^{(1)}_{[T,T(1+\beta)]}(aK) \le \beta I^{(1)}_{[0,T]}(aK)$ *and* $I^{(2)}_{[T,T(1+\beta)]}(K) \le \beta I^{(2)}_{[0,T]}(K)$.

Proof: (i) follows by a straightforward calculation. The bounds in (ii) and (iii) follow then in view of (i), equations (13) and (14) and since $m_2 \le m_1$.

(iv) Since $\overline{F}$ is decreasing it follows that $I^{(i)}_{[\alpha,\beta]}(\zeta)$ is bounded above by $(\beta-\alpha)\overline{F}(\zeta+m_i\alpha)$ and $I^{(i)}_{[0,\alpha]}(\zeta)$ is bounded below by $\alpha\overline{F}(\zeta+m_i\alpha)$. Combining these two estimates proves the statement. □

Since $\psi_\times \ge \psi_\vee$, we note that it will be sufficient to prove the following estimates:

$$\liminf_{K\to\infty} \frac{\psi_\wedge(aK,K)}{J(aK,K)} \ge 1, \tag{15}$$
$$\liminf_{K\to\infty} \frac{\psi_\vee(aK,K)}{H(aK,K)} \ge 1, \tag{16}$$

and

$$\limsup_{K\to\infty} \frac{\psi_\wedge(aK,K)}{J(aK,K)} \le 1, \tag{17}$$
$$\limsup_{K\to\infty} \frac{\psi_\times(aK,K)}{H(aK,K)} \le 1. \tag{18}$$

Lemma 2 *Suppose that F satisfies (3). Given $\eta > 0$, it holds for K large enough that*

$$P(M_{N^*}^{(j)} \leq a_j K, M_{[N^*+1,\infty)}^{(i)} > a_i K) \quad \sim \quad I_{[T,\infty]}^{(i)}(a_i K).$$

Proof It follows from Lemmas 6, 8, 7 and 9. □

Lemma 3 *Suppose F_I is long-tailed. Fix $0 < \eta < 1$. For K large enough it holds that*

$$P(M_{N^*}^{(i)} > a_i K) \quad \sim \quad I_{[0,T]}^{(i)}(a_i K), \quad i = 1, 2.$$

Proof By Theorem 2 of Foss et al. (2006) for sufficiently large K and small ϵ we have,

$$\begin{aligned}
P(M_{N^*}^{(i)} > a_i K) &\geq P(M_{[T(1-\epsilon)]}^{(i)} > a_i K, N^* > T(1-\epsilon)) \\
&\geq P(M_{[T(1-\epsilon)]}^{(i)} > a_i K) - P(N^* < T(1-\epsilon)) = P(M_{[T(1-\epsilon)]}^{(i)} > a_i K) - \mathrm{o}(I_{[0,T]}^{(i)}(a_i K)) \\
&\geq (1-\eta/2) I_{[0,T(1-\epsilon)]}^{(i)}(a_i K) = (1-\eta/2) I_{[0,T]}^{(i)}(a_i K) - (1-\eta/2) I_{[T(1-\epsilon),T]}^{(i)}(a_i K) \\
&\geq (1-\eta/2) I_{[0,T]}^{(i)}(a_i K) - (1-\eta/2)\epsilon I_{[0,T]}^{(i)}(a_i K) \geq (1-\eta) I_{[0,T]}^{(i)}(a_i K).
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(M_{N^*}^{(i)} > a_i K) &\leq P(M_{[T(1-\epsilon)]}^{(i)} > a_i K, N^* < T(1+\epsilon)) + P(N^* > T(1+\epsilon)) \\
&\leq P(M_{[T(1+\epsilon)]}^{(i)} > a_i K) + P(N^* > T(1+\epsilon)) = P(M_{[T(1+\epsilon)]}^{(i)} > a_i K) + \mathrm{o}(I_{[0,T]}^{(i)}(a_i K)) \\
&\leq (1+\eta/2) I_{[0,T(1+\epsilon)]}^{(i)}(a_i K) = (1+\eta/2) I_{[0,T]}^{(i)}(a_i K) + (1+\eta/2) I_{[T,T(1+\epsilon)]}^{(i)}(a_i K) \\
&\leq (1+\eta/2) I_{[0,T]}^{(i)}(a_i K) + (1+\eta/2)\epsilon I_{[0,T]}^{(i)}(a_i K) \leq (1+\eta) I_{[0,T]}^{(i)}(a_i K).
\end{aligned}$$

□

4.1 Lower bounds

Let

$$a_i = \begin{cases} a & \text{for } i = 1, \\ 1 & \text{for } i = 2. \end{cases}$$

Lemma 4 *Suppose $F_I \in \mathcal{S}$. Given $\eta > 0$, it holds for K large enough that*

$$P(M_{[N^*+1,\infty)}^{(i)} > a_i K) \quad \geq \quad (1-\eta) I_{[T,\infty]}^{(i)}(a_i K) - \eta I_{[0,T]}^{(i)}(a_i K).$$

Proof Fix $1 > \epsilon > 0$ and $0 < \delta < \epsilon$. By the law of large numbers it holds for K large enough that

$$P(S_{N^*}^{(i)} > -m_i(1+\epsilon)N^*, N^* < T(1+\epsilon)) > 1 - \delta. \tag{19}$$

Denote by $\widetilde{M}^{(i)}$ an independent copy of $M^{(i)}$. It follows by an application of Veraverbeke's theorem and Lemma 1 that for a_iK large enough it holds that

$$\begin{aligned}
&P(M^{(i)}_{[N^*+1,\infty)} > a_iK) \\
&\quad \geq \quad P(M^{(i)}_{[N^*+1,\infty)} > a_iK, S^{(i)}_{N^*} > -(m_i+\epsilon)N^*, N^* < T(1+\epsilon)) \\
&\quad \geq \quad P(\widetilde{M}^{(i)}_{[0,\infty)} > a_iK + m_i(1+\epsilon)^2T)P(S^{(i)}_{N^*} > -(m_i+\epsilon)N^*, N^* < T(1+\epsilon)) \\
&\quad \geq \quad (1-\delta)I^{(i)}_{[0,\infty]}(a_iK + m_i(1+3\epsilon)T) \\
&\quad \geq \quad (1-2\delta)\int_0^\infty \overline{F}(a_iK + m_i(1+3\epsilon)T + m_it)\mathrm{d}t \\
&\quad \geq \quad (1-2\delta)\int_T^\infty \overline{F}(a_iK + m_i(1+3\epsilon)t)\mathrm{d}t \\
&\quad = \quad (1-2\delta)\frac{m_i}{m_i(1+3\epsilon)}I^{(i)}_{[T(1+3\epsilon),\infty)}(a_iK) \\
&\quad \geq \quad (1-2\delta)\frac{1}{1+3\epsilon}\left(I^{(i)}_{[T,\infty]}(a_iK) - 3\epsilon I^{(i)}_{[0,T]}(a_iK)\right).
\end{aligned}$$

$\square$

Lemma 5 *Suppose $F_I \in \mathcal{S}$. For $i,j = 1,2$, given $\eta > 0$, it holds for K large enough that*

$$P(M^{(j)}_{N^*} \leq a_jK, M^{(i)}_{[N^*+1,\infty)} > a_iK) \quad \geq \quad I^{(i)}_{[T,\infty]}(a_iK) - \eta I^{(i)}_{[0,T]}(a_iK).$$

Proof As the previous Lemma, replacing (19) by

$$P(M^{(j)}_{N^*} \leq a_jK, S^{(i)}_{N^*} > -(m_i+\epsilon)N^*, N^* < T(1+\epsilon)) > 1-\delta.$$

$\square$

Lemma 6 *Suppose that F satisfies (3). Given $\eta > 0$, it holds for K large enough that*

$$P(M^{(i)}_{[N^*+1,\infty)} > a_iK) \geq (1-\eta)T\overline{F}(a_iK + m_iT) + (1-\eta)I^{(i)}_{[T,\infty]}(a_iK)$$

Proof Note that

$$P(M^{(i)}_{[N^*+1,\infty)} > a_iK) = P(S^{(i)}_{N^*+1} + \tilde{M}^{(i)}_{[0,\infty)} > a_iK),$$

where $\tilde{M}^{(i)}_{[0,\infty)}$ is all-time maximum from the independent random walk distributed as $S^{(i)}_n$. Moreover, for any $\epsilon > 0$,

$$\begin{aligned}
P(S^{(i)}_{N^*+1} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K) &\geq P(\Xi_{N^*+1} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i T_{N^*+1}, T(1+\epsilon) > N^* > T(1-\epsilon)) \\
&\geq P(\Xi_{[T(1-\epsilon)+1]} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i T_{[T(1+\epsilon)+1]}) \\
&\geq P(\Xi_{[T(1-\epsilon)+1]} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i T(1+\epsilon)^2 E\zeta) - P(N^* < T(1-\epsilon)) \\
&\quad - P(N^* > T(1+\epsilon)) - P(T_{[T(1+\epsilon)+1]} > (1+\epsilon)^2 E\zeta T) \\
&\geq P(\Xi_{[T(1-\epsilon)+1]} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i T(1+\epsilon)^2 E\zeta) - \mathrm{o}(I^{(i)}_{[T,\infty]}(a_i K)). \quad (20)
\end{aligned}$$

Further, for $1 < \alpha < 2$ and

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}$$

we derive

$$\begin{aligned}
&P(\Xi_{[T(1-\epsilon)+1]} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i T(1+\epsilon)^2 E\zeta) \\
&\quad = P\left(\frac{\Xi_{[T(1-\epsilon)+1]} - E\sigma T(1-\epsilon)}{(A/C_\alpha)^{1/\alpha} T^{1/\alpha}(1-\epsilon)^{1/\alpha}} (A/C_\alpha)^{1/\alpha} T^{1/\alpha}(1-\epsilon)^{1/\alpha} + \tilde{M}^{(i)}_{[0,\infty)} > (a_i+\kappa)K + m_i T \right),
\end{aligned}$$

where $\kappa = \epsilon(E\sigma - p_i E\zeta(2+\epsilon))(1-a)/(m_1 - m_2)$. If F satisfies (3) with $1 < \alpha < 2$, then by Theorem 4.5.2 of Whitt (2000) it is in a normal domain of attraction of the stable law $S_\alpha(1,1,0)$. Denoting $F_S(dx)$ the distribution of the stable law $S_\alpha(1,1,0)$ and writing $a_\alpha(K) = (A/C_\alpha)^{1/\alpha} T^{1/\alpha}(1-\epsilon)^{1/\alpha}$, we have

$$\begin{aligned}
&P\left(\frac{\Xi_{[T(1-\epsilon)+1]} - E\sigma T(1-\epsilon)}{a_\alpha(K)} a_\alpha(K) + \tilde{M}^{(i)}_{[0,\infty)} > (a_i+\kappa)K + m_i T \right) \\
&\quad = \int_{-\infty}^{\infty} P\left(x a_\alpha(K) + \tilde{M}^{(i)}_{[0,\infty)} > (a_i+\kappa)K + m_i T \right) dF_S(x) \\
&\quad \geq \int_{[(a_i+\kappa)K + m_i T]/a_\alpha(K)}^{\infty} P\left(x a_\alpha(K) + \tilde{M}^{(i)}_{[0,\infty)} > (a_i+\kappa)K + m_i T \right) dF_S(x) \\
&\quad + \int_{+\kappa[(a_i+\kappa)K + m_i T]/a_\alpha(K)}^{\kappa[(a_i+\kappa)K + m_i T]/a_\alpha(K)} P\left(x a_\alpha(K) + \tilde{M}^{(i)}_{[0,\infty)} > (a_i+\kappa)K + m_i T \right) dF_S(x).
\end{aligned}$$

Recall that

$$\overline{F}_S(x) \sim C_\alpha x^{-\alpha}, \quad F_S(-x) = \mathrm{o}(x^{-\alpha}) \quad \text{as } x \to \infty. \quad (21)$$

Hence using integration-by-parts or monotone density theorem, result of Embrechts and Veraverbeke (1982), for large K we have

$$
\begin{aligned}
&P\left(\frac{\Xi_{[T(1-\epsilon)+1]}-E\sigma T(1-\epsilon)}{a_\alpha(K)}a_\alpha(K)+\tilde{M}^{(i)}_{[0,\infty)}>(a_i+\kappa)K+m_iT\right)\\
&\quad\geq(1-\eta/2)[\overline{F}_S([(a_i+\kappa)K+m_iT]/a_\alpha(K))\\
&\quad+\frac{C_\alpha A}{m_i(\alpha-1)}\int_{-\kappa[(a_i+\kappa)K+m_iT]/a_\alpha(K)}^{\kappa[(a_i+\kappa)K+m_iT]/a_\alpha(K)}((a_i+\kappa)K+m_iT-xa_\alpha(K))^{-\alpha+1}x^{-\alpha-1}\,dx]\\
&\quad\geq(1-\eta/2)A[T(1-\epsilon)((a_i+\kappa)K+m_iT)^{-\alpha}\\
&\quad+\frac{A}{m_i(\alpha-1)}((1+\kappa)((a_i+\kappa)K+m_iT))^{-\alpha+1}(1-\overline{F}_S(\kappa((a_i+\kappa)K+m_iT)/a_\alpha(K))\\
&\quad-F_S(-\kappa((a_i+\kappa)K+m_iT)/a_\alpha(K)))]\\
&\quad\geq(1-\eta)AT(a_iK+m_iT)^{-\alpha}+(1-\eta)\frac{A}{m_i(\alpha-1)}(a_iK+m_iT)^{-\alpha+1},
\end{aligned}
$$

which completes the proof. □

Lemma 7 *Suppose that F satisfies (3). Given $\eta>0$, it holds for K large enough that*

$$P(M^{(j)}_{N^*}>a_jK,M^{(i)}_{[N^*+1,\infty)}>a_iK)\geq(1-\eta)T\overline{F}(a_jK+m_jT)=(1-\eta)T\overline{F}(a_iK+m_iT).$$

Proof We have

$$
\begin{aligned}
&P(M^{(j)}_{N^*}>a_jK,M^{(i)}_{[N^*+1,\infty)}>a_iK)\geq P(M^{(j)}_{N^*}>a_jK,M^{(i)}_{[N^*+1,\infty)}>a_iK,N^*>T(1-\epsilon))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,M^{(i)}_{[[T(1-\epsilon)],\infty)}>a_iK)-P(N^*<T(1-\epsilon))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,S^{(i)}_{[T(1-\epsilon)]}>a_iK)-\mathrm{o}(T\overline{F}(a_jK+m_jT))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,\Xi_{[T(1-\epsilon)]}>a_iK+p_iE\zeta T(1-\epsilon^2),T_{[T(1-\epsilon)]}<E\zeta T(1-\epsilon^2))\\
&\quad-\mathrm{o}(T\overline{F}(a_jK+m_jT))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,\Xi_{[T(1-\epsilon)]}-E\sigma T(1-\epsilon)>(1+\epsilon)(a_iK+m_iT)+\epsilon(1-\epsilon)p_iE\zeta T)\\
&\quad-P(T_{[T(1-\epsilon)]}>E\zeta T(1-\epsilon^2))-\mathrm{o}(T\overline{F}(a_jK+m_jT))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,\Xi_{[T(1-\epsilon)]}-E\sigma T(1-\epsilon)>(1+\epsilon)(a_jK+m_jT)+\epsilon p_iE\zeta T)\\
&\quad-\mathrm{o}(T\overline{F}(a_jK+m_jT))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,S^{(j)}_{[T(1-\epsilon)]}>(1+\epsilon)a_jK+\epsilon(p_i+p_j)E\zeta T-2\epsilon E\sigma T,T_{[T(1-\epsilon)]}>E\zeta T(1-\epsilon)^2)\\
&\quad-\mathrm{o}(T\overline{F}(a_jK+m_jT))\\
&\quad\geq P(M^{(j)}_{[T(1-\epsilon)]-1}>a_jK,S^{(j)}_{[T(1-\epsilon)]}>(1+\epsilon+\xi)a_jK)-\mathrm{o}(T\overline{F}(a_jK+m_jT))
\end{aligned}
$$

where $\xi = \epsilon((p_i + p_j)E\zeta - 2E\sigma)(1-a)/a_j(m_1 - m_2)$. Denote by $X_k^{(i)}$ the kth increment of the random walk $S_n^{(i)}$. Then

$$
\begin{aligned}
&P(M^{(j)}_{[T(1-\epsilon)]-1} > a_jK, S^{(j)}_{[T(1-\epsilon)]} > (1+\epsilon+\xi)a_jK) \\
&\quad \geq P(M^{(j)}_{[T(1-\epsilon)]-1} > a_jK, S^{(j)}_{[T(1-\epsilon)]} > (1+\epsilon+\xi)a_jK, X_{[T(1-\epsilon)]} < \epsilon a_jK) \\
&\quad \geq P(M^{(j)}_{[T(1-\epsilon)]-1} > a_jK, S^{(j)}_{[T(1-\epsilon)]-1} > (1+2\epsilon+\xi)a_jK, X_{[T(1-\epsilon)]} > -\epsilon a_jK) \\
&\quad = P(S^{(j)}_{[T(1-\epsilon)]-1} > (1+2\epsilon+\xi)a_jK, X_{[T(1-\epsilon)]} > -\epsilon a_jK) \\
&\quad \geq P(S^{(j)}_{[T(1-\epsilon)]-1} > (1+2\epsilon+\xi)a_jK) - P(X_{[T(1-\epsilon)]} < -\epsilon a_jK) \\
&\quad = P(S^{(j)}_{[T(1-\epsilon)]-1} > (1+2\epsilon+\xi)a_jK) - \mathrm{o}(T\overline{F}(a_jK + m_jT)).
\end{aligned}
$$

The assertion of the lemma follows now from the similar considerations as previously and given in the proof of Lemma 6:

$$
\begin{aligned}
&P(S^{(j)}_{[T(1-\epsilon)]-1} > (1+2\epsilon+\xi)a_jK) \\
&\quad \geq P\left((\Xi_{[T(1-\epsilon)]-1} - E\sigma T(1-\epsilon))/a_\alpha(K) > (1+\epsilon)(1+2\epsilon+\xi)(a_jK + m_jT)/a_\alpha(K)\right) \\
&\quad -P(T_{[T(1-\epsilon)]-1} > TE\zeta(1+\epsilon)^2) \\
&\quad \geq \overline{F}_S\left((1+\epsilon)(1+2\epsilon+\xi)(a_jK + m_jT)/a_\alpha(K)\right) - \mathrm{o}(T\overline{F}(a_jK + m_jT)) \\
&\quad \geq (1-\eta)\overline{F}_S\left((a_jK + m_jT)/a_\alpha(K)\right) \\
&\quad = (1-\eta)T\overline{F}(a_jK + m_jT).
\end{aligned}
$$

□

4.2 Upper bounds

Lemma 8 *Suppose that F satisfies (3). Given $\eta > 0$, it holds for K large enough that*

$$P(M^{(i)}_{[N^*+1,\infty)} > a_iK) \leq (1+\eta)T\overline{F}(a_iK + m_iT) + (1+\eta)I^{(i)}_{[T,\infty]}(a_iK)$$

Proof Similarly like in the proof of the lower bound given in Lemma 6, for large K and some $\kappa > 0$ we have,

$$
\begin{aligned}
&P(S^{(i)}_{N^*+1} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K) \le P(\Xi_{N^*+1} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i T_{N^*+1}, N^* < T(1+\epsilon)) \\
&\quad + P(N^* > T(1+\epsilon)) \\
&\quad \le P(\Xi_{[T(1+\epsilon)+1]} + \tilde{M}^{(i)}_{[0,\infty)} > a_i K + p_i \tilde{T}) + \mathrm{o}(I^{(i)}_{[T,\infty]}(a_i K)) \\
&\quad \le P\left(\frac{\Xi_{[T(1+\epsilon)+1]} - E\sigma T(1+\epsilon)}{b_\alpha(K)} b_\alpha(K) + \tilde{M}^{(i)}_{[0,\infty)} > (a_i - \kappa)K + m_i T \right) \\
&\quad + \mathrm{o}(I^{(i)}_{[T,\infty]}(a_i K)) \\
&\quad \le (1+\eta/2)[\overline{F}_S([(a_i-\kappa)K + m_i T]/b_\alpha(K)) \\
&\quad + \frac{C_\alpha A}{m_i(\alpha-1)} \int_{-\kappa[(a_i-\kappa)K+m_iT]/b_\alpha(K)}^{\kappa[(a_i-\kappa)K+m_iT]/b_\alpha(K)} ((a_i-\kappa)K + m_i T - x b_\alpha(K))^{-\alpha+1} x^{-\alpha-1}\, dx \\
&\quad + \frac{C_\alpha A}{m_i(\alpha-1)} \int_{\kappa[(a_i-\kappa)K+m_iT]/b_\alpha(K)}^{[(a_i-\kappa)K+m_iT]/b_\alpha(K)} ((a_i-\kappa)K + m_i T - x b_\alpha(K))^{-\alpha+1} x^{-\alpha-1}\, dx \\
&\quad + F_S(-\kappa[(a_i-\kappa)K + m_i T]/b_\alpha(K))],
\end{aligned}
$$

where $b_\alpha(K) = (A/C_\alpha)^{1/\alpha} T^{1/\alpha} (1+\epsilon)^{1/\alpha}$. The first two increments give required asymptotics since the second one can be estimated above by

$$
\begin{aligned}
&\frac{A}{m_i(\alpha-1)} \left((1-\kappa)\left((a_i-\kappa)K + m_i T\right)\right)^{-\alpha+1} \\
&\quad \left(1 - \overline{F}_S\left(\kappa((a_i-\kappa)K + m_i T)/b_\alpha(K)\right) - F_S\left(-\kappa((a_i-\kappa)K + m_i T)/b_\alpha(K)\right)\right).
\end{aligned}
$$

The last increment is $\mathrm{o}(I^{(i)}_{[T,\infty]}(a_i K))$ by (21). Finally, the third increment could be estimated in the following way,

$$
\begin{aligned}
&\frac{C_\alpha A}{m_i(\alpha-1)} \int_{\kappa[(a_i-\kappa)K+m_iT]/b_\alpha(K)}^{[(a_i-\kappa)K+m_iT]/b_\alpha(K)} ((a_i-\kappa)K + m_i T - x b_\alpha(K))^{-\alpha+1} x^{-\alpha-1}\, dx \\
&\quad = b_\alpha(K)^\alpha \frac{C_\alpha A}{m_i(\alpha-1)} \int_{\kappa[(a_i-\kappa)K+m_iT]}^{[(a_i-\kappa)K+m_iT]} ((a_i-\kappa)K + m_i T - w)^{-\alpha+1} w^{-\alpha-1}\, dw \\
&\quad = ((a_i-\kappa)K + m_i T)^{-2\alpha+1} T \frac{A^2(1+\epsilon)}{m_i(\alpha-1)} \int_\kappa^1 (1-t)^{-\alpha+1} t^{-\alpha-1}\, dt \\
&\quad = \mathrm{O}(K^{-2\alpha+2}) = \mathrm{o}(K^{-\alpha+1}) = \mathrm{o}(I^{(i)}_{[T,\infty]}(a_i K)).
\end{aligned}
$$

□

Lemma 9 *Suppose that F satisfies (3). Given $\eta > 0$, it holds for K large enough that*

$$
P(M^{(j)}_{N^*} > a_j K, M^{(i)}_{[N^*+1,\infty)} > a_i K) \le (1+\eta) T \overline{F}(a_j K + m_j T) = (1+\eta) T \overline{F}(a_i K + m_i T)
$$

Proof Similarly like in the proof of the Lemma 7 for large K we have,

$$\begin{aligned}
&P(M^{(j)}_{N^*} > a_jK, M^{(i)}_{[N^*+1,\infty)} > a_iK) \\
&\quad \le P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, M^{(i)}_{[[T(1+\epsilon)],\infty)} > a_iK) + P(N^* > T(1+\epsilon)) \\
&\quad \le P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, \Xi_{[T(1+\epsilon)]} > a_iK + p_iE\zeta T(1-\epsilon^2), T_{[T(1+\epsilon)]} > E\zeta T(1-\epsilon^2)) \\
&\quad +\mathrm{o}(T\overline{F}(a_jK + m_jT)) \\
&\quad \le P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, \Xi_{[T(1+\epsilon)]} - E\sigma T(1+\epsilon) > (1-\epsilon)(a_iK + m_iT) - 2\epsilon E\sigma T) \\
&\quad +P(T_{[T(1+\epsilon)]} < E\zeta T(1-\epsilon^2)) + \mathrm{o}(T\overline{F}(a_jK + m_jT)) \\
&\quad \le P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, S^{(j)}_{[T(1+\epsilon)]} > (1-\epsilon)a_jK - \epsilon(1+3\epsilon)p_jE\zeta T - 2\epsilon E\sigma T, \\
&\quad T_{[T(1+\epsilon)]} < E\zeta T(1+\epsilon)^2) + P(T_{[T(1+\epsilon)]} > E\zeta T(1+\epsilon)^2) + \mathrm{o}(T\overline{F}(a_jK + m_jT)) \\
&\quad \le P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, S^{(j)}_{[T(1+\epsilon)]} > (1-\epsilon-\chi)a_jK) + \mathrm{o}(T\overline{F}(a_jK + m_jT)),
\end{aligned}$$

where $\chi = \epsilon[(1+3\epsilon)E\zeta + 2E\sigma](1-a)/a_j(m_1 - m_2)$. Then

$$\begin{aligned}
&P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, S^{(j)}_{[T(1+\epsilon)]} > (1-\epsilon-\chi)a_jK) \\
&\quad \le P(S^{(j)}_{[T(1+\epsilon)]-1} > (1-2\epsilon-\chi)a_jK) + P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK, X_{[T(1+\epsilon)]} > \epsilon a_jK) \\
&\quad \le P(S^{(j)}_{[T(1+\epsilon)]-1} > (1-2\epsilon-\chi)a_jK) + P(M^{(j)}_{[T(1+\epsilon)]-1} > a_jK)P(X_{[T(1+\epsilon)]} > \epsilon a_jK) \\
&\quad \le P(S^{(j)}_{[T(1+\epsilon)]-1} > (1-2\epsilon-\chi)a_jK) + \mathrm{O}(K^{-2\alpha+1}) \\
&\quad = P(S^{(j)}_{[T(1+\epsilon)]-1} > (1-2\epsilon-\chi)a_jK) + \mathrm{o}(T\overline{F}(a_jK + m_jT))
\end{aligned}$$

and

$$\begin{aligned}
&P(S^{(j)}_{[T(1+\epsilon)]-1} > (1-2\epsilon-\chi)a_jK) \\
&\quad \le P\left((\Xi_{[T(1+\epsilon)]-1} - E\sigma T(1+\epsilon))/b_\alpha(K) > (1-\epsilon)(1-2\epsilon\chi)(a_jK + m_jT)/b_\alpha(K)\right) \\
&\quad +P(T_{[T(1-\epsilon)]-1} < TE\zeta(1-\epsilon^2)) \\
&\quad \le \overline{F}_S\left((1-\epsilon)(1-2\epsilon-\chi)(a_jK + m_jT)/b_\alpha(K)\right) + \mathrm{o}(T\overline{F}(a_jK + m_jT)) \\
&\quad \ge (1-\eta)\overline{F}_S\left((a_jK + m_jT)/b_\alpha(K)\right) \\
&\quad = (1-\eta)T\overline{F}(a_jK + m_jT),
\end{aligned}$$

which completes the proof. □

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